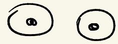
Elliptic Functions (Iwaniec 1.2)

$f: \mathbb{C} \to \mathbb{C}$ holomorphic or meromorphic
$\quad \Omega \to \mathbb{C}$

holomorphic: differentiable as a complex fct at any $z_0 \in U \subseteq \mathbb{C}$
$\Rightarrow$ infinitnely differentiable at every point $z_0 \in U \subseteq \mathbb{C}$ and analytic (Equal to its Taylor series at z_0)

meromorphic: holomorphic at every point $z_0 \in U \subseteq \mathbb{C}$, except for a set of isolated points, which are poles of the function



Then, f is meromorphic on U if at each $z_0 \in U$, f or $\frac{1}{f}$ is holomorphic

Important notion: f has a pole at $z_0 \iff \frac{1}{f}$ has a zero at z_0

Then, for f meromorphic on U, and any $z_0 \in U$, $\exists! n \in \mathbb{Z}$ st $(z-z_0)^n f(z)$ is holomorphic & non zero in a neighborhood of z_0

$n > 0$ pole of order n at z_0 (order or multiplicity)
$n < 0$ zero of order $|n|$ at z_0
$n = 0$ f is holomorphic & non zero at z_0

$$\boxed{n = \operatorname{ord}_{z_0}(f)}$$

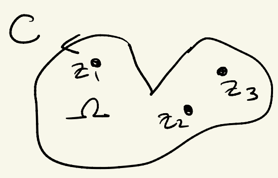
Also, if f is meromorphic on U, then for any $z_0 \in U$,

$$f(z) = \sum_{k \geq -n} a_k (z-z_0)^k, \quad (a_{-n} \neq 0) \quad \text{(Laurent series)}$$

$n > 0$ pole of order n at z_0
$n \leq 0$ zero of order n at z_0

Cauchy's thm
$$\oint_C f(z)\,dz = \sum_{i=1}^{k} \operatorname*{res}_{z=z_i} f$$

where



$f(z)$ is meromorphic on Ω with poles at $z_1, \to z_k \in \Omega$

<u>residues</u> $\operatorname{res}_{z=z_0} f$ is the coefficient a_{-1} in the Laurent series $f(z) = \sum_{k \geq -n} a_k (z-z_0)^k$

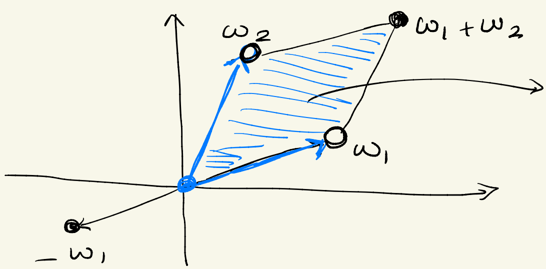
Then $\operatorname{res}_{z=z_0} f = \lim_{z \to z_0} (z-z_0) f(z)$ if z_0 is a <u>simple</u> pole

and in general for a pole of <u>order n</u> $\operatorname{res}_{z=z_0} f = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right]$

<u>Remark</u> The residue at a simple pole is never 0, but the residue at a double pole can be as $f(z) = \frac{1}{z}$

<u>Elliptic Functions</u>

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1, \omega_2 \in \mathbb{C}$ ~~st~~ $\mathbb{C} = \mathbb{R}\omega_1 + \mathbb{R}\omega_2$.
Then Λ is a discrete (inside $\mathbb{C}$) <u>free abelian subgroup</u> of $\mathbb{C}$ <u>of rank 2</u>, which is called a lattice



$\mathcal{F}$ = <u>fundamendal domain</u> for Λ which is a set of representatives for the quotient group

$$\mathbb{C}/\Lambda = \{ \alpha + \Lambda : \alpha \in \mathcal{F} \}$$

$$\mathcal{F} = \{ t_1\omega_1 + t_2\omega_2 : 0 \leq t_1, t_2 < 1 \}$$

$$\mathcal{F} = \{ t_1\omega_1 + t_2\omega_2 + \mu : 0 \leq t_1, t_2 < 1, \mu \in \mathbb{C} \}$$

<u>Def</u> A function $f: \mathbb{C} \to \mathbb{C}$ is <u>elliptic</u> <u>for the lattice Λ</u> if

(1) f is meromorphic on $\mathbb{C}$

(2) f is periodic with periods in Λ <u>ie</u> $f(u+\omega) = f(u)$ $\forall \omega \in \mathbb{C}$

$\iff f(u+\omega_1) = f(u+\omega_2) = f(u)$

<u>Remarks</u> ① Then, f is a meromorphic function on the torus $\mathbb{C}/\Lambda$  → Riemann surface (of genus 1)

② If f, g are elliptic, then so are $f \pm g$, fg, and $\frac{f}{g}$, and the elliptic functions form a field $\mathbb{C}(\Lambda) \supseteq \mathbb{C}$ (constant functions)

This is called <u>the field of elliptic functions of Λ</u>

Can we find some elliptic fcts?

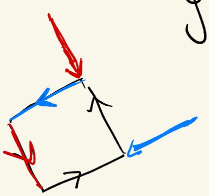
① $f \in \mathbb{C}(\Lambda)$ holomorphic: holomorphic and bounded on $\mathbb{C}$

$\Rightarrow f \in \mathbb{C}$ by Liouville thm

② $f \in \mathbb{C}(\Lambda)$ meromorphic: let $\mathcal{F}$ be fundamental region for Λ <u>st</u> the boundary of $\mathcal{F}$ does not contain any pole of f (there are only finitely many poles of f in $\mathcal{F}$ since this is a bounded region of $\mathbb{C}$)

Then using <u>Cauchy's thm</u>

$$\oint_C f \, dz = 0 = \sum_{z \in \mathcal{F}} \mathrm{res}_f(z) \qquad (*)$$



by periodicity

$\Rightarrow$ there is no function $f \in \mathbb{C}(\Lambda)$ with exactly one simple pole

<u>Other application of</u> <u>Cauchy's thm</u>

① $$\sum_{z \,(\mathrm{mod}\, \Lambda)} \mathrm{ord}_z(f) = 0$$

$z \in \mathbb{C}/\Lambda$ or $z \in \mathcal{F}$

[Integrate $\frac{f'(u)}{f(u)}$ over C

or apply $(*)$ to $\frac{f'(u)}{f(u)}$]

② $\sum_{z (\text{mod } \Lambda)} \text{ord}_z(f)\, z \equiv 0 \mod \Lambda$ [integrating $u \frac{f'}{f}$ over C]

[addition in the abelian group $\mathbb{C}/\Lambda$]

Remark

① is reminescent of : Principal divisor have degree 0

$$\text{Div}(\mathbb{C}/\Lambda) = \Big\{ \underbrace{\sum}_{\text{finite sum}} n_z\, z : n_z \in \mathbb{Z},\ z \in \mathbb{C}/\Lambda \Big\}$$

$$\text{Div}^0(\mathbb{C}/\Lambda) = \Big\{ D = \sum_z n_z\, z \in \text{Div}(\mathbb{C}/\Lambda) : \deg(D) := \sum n_z = 0 \Big\}$$

For $f \in \mathbb{C}(\Lambda)$, we define $\text{div}(f) = \underbrace{\sum_{z \text{ mod } \Lambda} \text{ord}_z(f)\, z}_{\text{called a principal divisor}}$.

② $\sum n_z\, z$ is a principal divisor $\Longleftrightarrow \sum_{z \text{ mod } \Lambda} n_z = 0$

and $\underbrace{\sum_{z \text{ mod } \Lambda} n_z\, z \equiv 0\ (\Lambda)}_{\text{addition on } T = \mathbb{C}/\Lambda \text{ induced by addition in } \mathbb{C}}$

Reminescent of

E elliptic curve (projective curve of genus 1) & $D = \sum_P n_P\, P \in \text{Div}(E)$

Then D is principal $\Longleftrightarrow \sum n_P = 0$

and $\sum n_P\, P = 0$ (addition on E).

Def Let $f \in \mathbb{C}(\Lambda)$. Then

$$\operatorname{ord}(f) = \sum_{z \bmod \Lambda} \max\left(\operatorname{ord}_z(f), 0\right)$$
$$= -\sum_{z \bmod \Lambda} \min\left(\operatorname{ord}_z(f), 0\right)$$

Equal because of 1

Lemma Let $f \in \mathbb{C}(\Lambda)$, and $c \in \mathbb{C}$. Then $f(u) = c$ has exactly $\operatorname{ord}(f)$ solutions in $\mathbb{C}/\Lambda$ (counted with multiplicity).

proof By definition, the number of solutions of $f(u) = 0$ in $\mathbb{C}/\Lambda$ is $\operatorname{ord}(f)$.

Now for $c \neq 0$, by 1

$$0 = 0 - 0 = \sum_{z \bmod \Lambda} \operatorname{ord}_z(f) - \sum_{z \bmod \Lambda} \operatorname{ord}_z(f-c)$$
$$= \sum_{z \bmod \Lambda} \max(0, \operatorname{ord}_z(f)) - \sum_{z \bmod \Lambda} \max(0, \operatorname{ord}_z(f-c)),$$

ie $\operatorname{ord}(f) = \operatorname{ord}(f-c)$

since the poles of f & $f-c$ are at the same points and of the same order. This proves the result $\square$

Simplest elliptic function double pole at $u=0$ (with zero residue)

$$\wp(u) = \frac{1}{u^2} + \sum_{\omega \in \Lambda}{}' \frac{1}{(u-\omega)^2} - \frac{1}{\omega^2}$$

[where ′ indicates that $\omega = 0$ is not there]

To show
1. meromorphic on $\mathbb{C}$
2. Periodic for Λ
3. (easy) The only pole is a pole of order 2 at any $\omega \in \Lambda$ ie $\wp(u)$ has one pole of order 2 at $u=0$ as a fct on $\mathbb{C}/\Lambda$
4. (easy) $\wp(u) = \wp(-u)$ (even fct)

① we show that $\wp(u)$ cvgs absolutely & uniformly on any compact subset of $\mathbb{C} - \Lambda$

② Then $\wp'(u) = \frac{-2}{u^3} + \sum_{\omega \in \Lambda}' \frac{-2}{(u-\omega)^3} = -2 \sum_{\omega \in \Lambda} \frac{1}{(u-\omega)^3}$

which is obviously periodic mod Λ

$\Rightarrow (\wp(u+\omega_1) - \wp(u))' = \wp'(u+\omega_1) - \wp'(u) = 0$

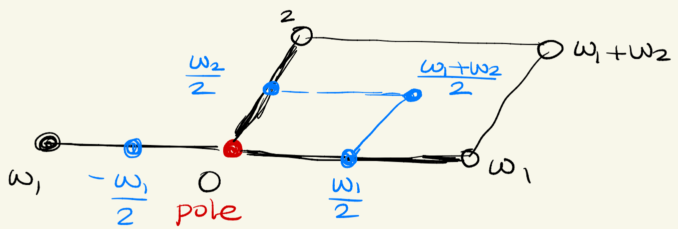
$\Rightarrow \wp(u+\omega_1) - \wp(u) = c$. Using $u = -\frac{1}{2}\omega_1$ & the fact that $\wp(u)$ is even $\Rightarrow c = 0$.

Lemma

$f(u) = \wp(u) - \wp(z)$ has 2 simple zeroes at $u = \pm z$ if $z \not\equiv -z \ (\Lambda)$ and a double zero if $z \equiv -z \ (\Lambda)$ (and $z \neq 0$)

Proof Since $\deg(\wp) = 2$, and $\wp(\pm z) - \wp(z) = 0$, there are 2 simple poles if $z \not\equiv -z \ (\Lambda)$.
If $z \equiv -z \ (\Lambda)$, then $f(z) = 0$ and $f'(z) = \wp'(z)$ is also 0 since $\wp'(u)$ is odd. ∎

2 torsion points of $\mathbb{C}/\Lambda$ $z \in \mathbb{C}/\Lambda \Leftrightarrow z \equiv -z \ (\Lambda)$
$\Leftrightarrow 2z \equiv 0 \ (\Lambda)$



$T = \{0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\}$

torsion points of order 2 in the abelian group $\mathbb{C}/\Lambda$

$\Rightarrow$ there are exactly 3 points of $\mathbb{C}/\Lambda$ st $\wp(u) - \wp(z)$ has a double zero $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$

The 3 complex numbers

$$e_1 = \wp\left(\frac{\omega_1}{2}\right), \quad e_2 = \wp\left(\frac{\omega_2}{2}\right), \quad e_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$$

are distinct since we dont have $\omega_1 \equiv \pm \omega_2 \mod \Lambda$ etc

Thm $\mathbb{C}(\wp, \wp') = \mathbb{C}(\Lambda)$ Rmk $\mathbb{C}(\wp, \wp') \subseteq \mathbb{C}(\Lambda)$

proof First suppose that $f \in \mathbb{C}(\Lambda)$ is $\boxed{\text{even}}$, with zeroes $z_1, \dots, z_m$ of order $a_1, \dots, a_m$
poles $p_1, \dots, p_n$ —— $b_1, \dots, b_n$

Let
$$g(u) = \frac{\prod_i{}' \left(\wp(u) - \wp(z_i)\right)^{\mathrm{ord}_f(z_i)}}{\prod_j \left(\wp(u) - \wp(p_j)\right)^{|\mathrm{ord}_f(p_j)|}}$$

where the prime means: ① we take only one of $\pm z_i$ for each i (and one of $\pm p_j$ for each j) when $z_i \not\equiv -z_i \ (\Lambda)$
② we replace $\mathrm{ord}_f(z_i)$ by $\mathrm{ord}_f(z_i)/2$ if $z_i \equiv -z_i \ (\Lambda)$

If $z_i \notin \{\omega_1, \omega_2, \frac{\omega_1+\omega_2}{2}\}$, then $\mathrm{ord}_{z_i}(g) = \mathrm{ord}_{z_i}(f)$.
If $z_i \in \{\omega_1, \omega_2, \frac{\omega_1+\omega_2}{2}\}$, then we show $\mathrm{ord}_{z_i}(f)$ is even

Laurent series $f(z) = \sum_{k \geq m} a_m (z - z_i)^m + \text{hot}$

$\Rightarrow f(z+z_i) = \sum_{k \geq m} a_m z^m + \text{hot}$, $\quad f(-z+z_i) = \sum_{k \geq m} a_m (-z)^m + \text{hot}$.

Also $f(-z+z_i) \underset{\text{even}}{=} f(z - z_i) = f(z - z_i + 2z_i) = f(z+z_i)$
periodic
& $2z_i \equiv 0 \ (\Lambda)$

and $\sum a_m (z^m) + \text{hot} = \sum a_m (-z)^m + \text{hot}$
$\Rightarrow \boxed{m \text{ is even}}$

Then, we have that g & f have exactly the same zeroes & same poles $\Rightarrow f = cg$. & $f \in \mathbb{C}(\wp)$.

Now, if f is any fct in $\mathbb{C}(\Lambda)$, then

$$g_1 = \frac{f(u) + f(-u)}{2}, \quad g_2 = \frac{f(u) - f(-u)}{2\wp'(u)}$$ are both even,

and $f = g_1(u) + \wp'(u) g_2(u) \Rightarrow f \in \mathbb{C}(\wp, \wp')$ ∎

Thm $\wp'(u)^2 = 4(\wp(u)-e_1)(\wp(u)-e_2)(\wp(u)-e_3)$

where $e_1 = \wp\left(\frac{\omega_1}{2}\right)$, $e_2 = \wp\left(\frac{\omega_2}{2}\right)$, $e_3 = \wp\left(\underbrace{\frac{\omega_1+\omega_2}{2}}_{\omega_3}\right)$

proof Let $f(u) = \dfrac{\wp'(u)^2}{(\wp(u)-e_1)(\wp(u)-e_2)(\wp(u)-e_3)}$

$\wp'(u)$: triple pole at $u=0$ and simple zero at $\omega_i/2$

double pole at $u=0$ and double zero at $\frac{\omega_i}{2}$

Note The zeroes of $\wp(u)$ are not easy to write down.

Then $f(u)$ has no zero or no pole $\Rightarrow f(u) \in \mathbb{C}$

To find the constant, compare the Laurent expansions of $\wp(u)$ & $\wp'(u)$ at $u=0$

$$\wp(u) = \frac{1}{u^2} + \sum_{\omega\in\Lambda}' \frac{1}{(\omega-u)^2} - \frac{1}{\omega^2}$$

$$= \frac{1}{u^2} + \sum_{\omega\in\Lambda}' \frac{1}{\omega^2}\left[\sum 1 + \frac{u}{\omega} + \left(\frac{u}{\omega}\right)^2 + \cdots\right]^2 - \frac{1}{\omega^2}$$

$$= \frac{1}{u^2} + \sum_{\omega\in\Lambda} \frac{1}{\omega^2}\left[1 + 2\frac{u}{\omega} + 3\frac{u^2}{\omega^2} + \cdots\right] - \frac{1}{\omega^2}$$

$$= \frac{1}{u^2} + \sum_{\omega\in\Lambda} 2\frac{u}{\omega^3} + 3\frac{u^2}{\omega^4} + 4\frac{u^3}{\omega^5} + \frac{5u^4}{\omega^6} + \cdots$$

$$= \frac{1}{u^2} + 2u\,G_3(\Lambda) + 3u^2 G_4(\Lambda) + \cdots$$

where $G_k(\Lambda) = \sum_{\omega \in \Lambda}{}' \frac{1}{\omega^k} = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\omega_1 + n\omega_2)^k}$

$\underline{k \geq 3}$

It is easy to see that $G_k(\Lambda) = 0$ for k odd, which makes sense since $\wp(u)$ is <u>even</u>

Then $\wp'(u) = \frac{-2}{u^3} + 6uG_4(\Lambda) + 20u^4 G_6(\Lambda) + \dots$

Then $\wp'(u)^2 = c(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3)$

$\Rightarrow \frac{4}{u^6} = c\left[\frac{1}{u^2}\right]^3 \Rightarrow c = 4$, and

$\wp'(u)^3 = 4(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3)$

Comparing the Taylor expansion of $\wp'(u)$, $\wp(u)$, $\wp(u)^2$, $\wp(u)^3$, we get

$\wp'(u)^3 = 4\wp(u)^3 - \underbrace{60G_4(\Lambda)}_{g_2(\Lambda)}\wp(u) - \underbrace{140G_6(\Lambda)}_{g_4(\Lambda)}$

<u>Thm</u> (complex Uniformization)

$\mathbb{C}/\Lambda \longrightarrow \{(x,y) \in \mathbb{C}^2 : y^2 = 4\cdot$
$y^2 = 4x^3 - g_2(\Lambda)x - g_4(\Lambda)\}$

$z \neq 0 \longmapsto (\wp(z), \wp'(z))$

is a bijection, or the map

$$\begin{aligned}
\mathbb{C}/\Lambda &\longrightarrow \{ [(x,y,z)] \in \mathbb{P}^2(\mathbb{C}) : \\
&\qquad y^2 z = 4x^3 - g_2(\Lambda) x z^2 - g_4(\Lambda) z^3 \} \\
z \neq 0 &\longmapsto [\wp(z), \wp'(z), 1] \\
0 &\longmapsto [0,1,0]
\end{aligned}$$

is an <u>analytic</u> <u>1-1 correspondance</u> between $\mathbb{C}/\Lambda$ and the points of $y^2 = 4x^3 - g_2 x - g_3$ in $\mathbb{P}^2(\mathbb{C})$.

<u>Field of elliptic functions</u>:

$$\{\text{Meromorphic fcts on } \mathbb{C}/\Lambda\} = \mathbb{C}(\wp, \wp')$$

<u>Elliptic Function field</u> (over $\mathbb{C}$)

$\{$rational fcts defined on $E: y^2 = x^3 + ax + b.\}$

$$= \mathbb{C}[x,y] \big/ (y^2 - x^3 - ax - b)$$

$$= \mathbb{C}(x)\left[\sqrt{x^3 + ax + b}\right] \simeq \mathbb{C}(\wp, \wp')$$